In the paper **Functional Pearl: F for Functor** from ICPF '12, the concept of a **bifunctor** is introduced quickly and somewhat confusingly. Herein, as Neil Gaiman wrote in Good Omens, "the text will be slowed down to allow the sleight of hand to be followed."

1 Bifunctors

A **bifunctor** is a two-argument object, here denoted $— \otimes — \in \mathcal{E}^{\mathcal{C}\times\mathcal{D}}$, which

- Sends an object $C \times D \in \mathcal{C} \times \mathcal{D}$ to an object $C \otimes D \in \mathcal{E}$ and a morphism $f \times g \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}(C \times D, C' \times D')$ to a morphism $f \otimes g \in \mathrm{Hom}_{\mathcal{E}}(C \otimes D, C' \otimes D')$.
- Preserves identities ($id_C \otimes id_D = id_{C\otimes D}$) and composition: $(f' \circ f) \otimes (g' \circ g) = (f' \otimes g') \circ (f \otimes g)$.
- Has a $\mathcal{C}$-object-indexed collection of *functors* obtained by partial application on the left: a $L^{\otimes}_C = (C \otimes —) \in \mathcal{E}^{\mathcal{D}}$ for each object $C \in \mathcal{C}$, and a $\mathcal{D}$-object-indexed collection from the right: a $R^{\otimes}_D(— \otimes D) \in \mathcal{E}^{\mathcal{C}}$ for each object $D \in \mathcal{D}$.[1]

The question arises: if we have two collections of functors, $\{L^{\otimes}_C \in \mathcal{E}^{\mathcal{D}} \mid C \in \mathcal{C}\}$ and $\{R^{\otimes}_D \in \mathcal{E}^{\mathcal{C}} \mid D \in \mathcal{D}\}$, can we stitch them together to make a bifunctor? Looking at the object component of our purported bifunctor $\otimes$, we see that $C \otimes D$ has two possible definitions: $L^{\otimes}_C D$ and $R^{\otimes}_D C$. These must be equal in order for $\otimes$ to be well-defined:

$$L^{\otimes}_C D = R^{\otimes}_D C \qquad (\forall_{C\in\mathcal{C},D\in\mathcal{D}}.\text{diagram} \in \mathcal{E}) \tag{1}$$

What of the morphism map? Given $f \in \mathrm{Hom}_{\mathcal{C}}(C, C')$ and $g \in \mathrm{Hom}_{\mathcal{D}}(D, D')$, $f \otimes g$ expands in one of two ways, as indicated by the following diagram. These, too, must be equal for the definition to make sense. Note that (1) allows us to label the vertices of this diagram in an unambiguous, familiar syntax.

$$\begin{tikzcd}
C \otimes D \arrow[r, "R^{\otimes}_D f"] \arrow[d, "L^{\otimes}_C g"'] \arrow[dr, phantom, "\circ"] & C' \otimes D \arrow[d, "L^{\otimes}_{C'} g"] \\
C \otimes D' \arrow[r, "R^{\otimes}_{D'} f"'] & C' \otimes D'
\end{tikzcd} \qquad (\forall_{C,C',f\in\mathcal{C};D,D',g\in\mathcal{D}}.\text{diagram} \in \mathcal{E}) \tag{2}$$

Let us check that any $\{L^{\otimes}_C \mid C\}$ and $\{R^{\otimes}_D \mid D\}$ which satisfy (1) and (2) in fact give rise to a bifunctor. We have our object and morphism maps and purported partial-applications already, all that remains to be seen is preservation of identity and composition. Identity is easy:

$$id_C \otimes id_D = L^{\otimes}_C id_D \circ R^{\otimes}_D id_C = id_{L^{\otimes}_C D} \circ id_{R^{\otimes}_D C} = id \tag{3}$$

$$= R^{\otimes}_D id_C \circ L^{\otimes}_C id_D = id_{R^{\otimes}_D C} \circ id_{L^{\otimes}_C D} = id \tag{4}$$

where (3) is the right-then-down path and (4) is the down-then-right path in (2). Note that we did *not* need to assume anything to get this other than that $L^{\otimes}_C$ and $R^{\otimes}_D$ were functors. Composition unpacks a bit: $(f' \otimes g') \circ (f \otimes g)$ (for $f' \in \mathrm{Hom}_{\mathcal{C}}(C', C'')$ and $g' \in \mathrm{Hom}_{\mathcal{D}}(D', D'')$ in addition to f and g as above) has many possible meanings, each of which is a path through this diagram:

$$\begin{tikzcd}
C \otimes D \arrow[r, "R^{\otimes}_D f"] \arrow[d, "L^{\otimes}_C g"'] & C' \otimes D \arrow[r, "R^{\otimes}_D f'"] \arrow[d, "L^{\otimes}_{C'} g"'] & C'' \otimes D \arrow[d, "L^{\otimes}_{C''} g"] \\
C \otimes D' \arrow[r, "R^{\otimes}_{D'} f"] \arrow[d, "L^{\otimes}_C g'"'] & C' \otimes D' \arrow[r, "R^{\otimes}_{D'} f'"] \arrow[d, "L^{\otimes}_{C'} g'"'] & C'' \otimes D' \arrow[d, "L^{\otimes}_{C''} g'"] \\
C \otimes D'' \arrow[r, "R^{\otimes}_{D''} f"] & C' \otimes D'' \arrow[r, "R^{\otimes}_{D''} f'"] & C'' \otimes D''
\end{tikzcd} \qquad (\forall_{C,C',C'',f,f'\in\mathcal{C};D,D',D'',g,g'\in\mathcal{D}}.\text{diagram} \in \mathcal{E}) \tag{5}$$

However, repeated application of (2) lets us see that all paths through this diagram are equal! Moreover, the horizontal and vertical paths correctly compose because each $L^{\otimes}_x$ and $R^{\otimes}_x$ are functors; that is, along the top, for example: $R^{\otimes}_D f' \circ R^{\otimes}_D f = R^{\otimes}_D (f' \circ f)$. We can see that the two possible expansions of $(f' \circ f) \otimes (g' \circ g)$ (the two paths of (2) with the compositions in as the functions) are (respectively) equal to the two outermost paths of (5) and therefore to each other.

F for Functor arrives at these conclusions in the reverse order. That is, it *defines* $f \otimes g$ as (in the notation of this document) $L^{\otimes}_{C'} g \circ R^{\otimes}_D f$ (the top-right path of (2)). Then it considers identity and composition, arriving (implicitly) at (5). "Lambert" computes the right-right-down-down and right-down-right-down paths, in accordance with the bias of the definition and concludes that the top-right rectangle of (5) must commute, thereby *deriving* (2).

[1] Of course, there are also functor families indexed by arrows, which might be designated $\tilde{L}^{\otimes}_f = (f \otimes —) \in \mathcal{E}^{\mathcal{D}}$ for each $f \in \mathcal{C}$. However, these bring no new degrees of freedom to the table, as $\tilde{L}^{\otimes}_f(D) = f \otimes D = R^{\otimes}_D f$ and $\tilde{L}^{\otimes}_f(g) = f \otimes g$.

2 Natural Transformations

We have used the exponential notation $\mathcal{E}^{\mathcal{C}}$ above possibly somewhat prematurely. While we could simply define functor categories as discrete, let us instead not. The objects of such a thing we take to be functors from $\mathcal{C}$ to $\mathcal{E}$. We will *recover* the definition of morphisms in this category by considering a particular bifunctor (later in the paper denoted $\star$) in $\mathcal{E}^{\mathcal{E}^{\mathcal{C}} \times \mathcal{C}}$. This bifunctor's partial application views are $\{L^\star_F \mid F \in \mathcal{E}^{\mathcal{C}}\}$ (denoted F— in the paper) and $\{R^\star_C \mid C \in \mathcal{C}\}$ (denoted $—A$ in the paper). $L^\star_F$ behaves as the functor F: it sends $C \in \mathcal{C}$ to $FC \in \mathcal{E}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(C, C')$ to $Ff \in \mathrm{Hom}_{\mathcal{E}}(FC, FC')$. This relies only on the objects of $\mathcal{E}^{\mathcal{C}}$ and so is uninteresting. What about $R^\star_C$? If it is to be a functor, then:

- The object component of $R^\star_C$ takes a functor $F \in \mathcal{E}^{\mathcal{C}}$ to an object $FC \in \mathcal{E}$.
- The morphism component of $R^\star_C$ takes a morphism $\alpha \in \mathrm{Hom}_{\mathcal{E}^{\mathcal{C}}}(F, G)$ to a morphism $R^\star_C\alpha \in \mathrm{Hom}_{\mathcal{E}}(R^\star_C F, R^\star_C G) = \mathrm{Hom}_{\mathcal{E}}(FC, GC)$.
- It must map the identity morphism to an identity morphism: $R^\star_C id_F = id_{FC}$.
- It must preserve composition: for any $\alpha \in \mathrm{Hom}_{\mathcal{E}^{\mathcal{C}}}(F, G)$ and $\alpha' \in \mathrm{Hom}_{\mathcal{E}^{\mathcal{C}}}(G, H)$, we must have that $R^\star_C(\alpha' \circ \alpha) = (R^\star_C\alpha') \circ (R^\star_C\alpha)$.
- Furthermore, if $\star$ is to be a bifunctor, then (2) must hold (this diagram is over all $F, G \in \mathcal{E}^{\mathcal{C}}$, $\alpha \in \mathrm{Hom}_{\mathcal{E}^{\mathcal{C}}}(F, G)$, $C, C' \in \mathcal{C}$, and $f \in \mathrm{Hom}_{\mathcal{C}}(C, C')$; it ultimately takes place in $\mathcal{E}$):

$$\begin{tikzcd} F \star C \arrow[r, "R^\star_C\alpha"] \arrow[d, "L^\star_F f"'] \arrow[dr, phantom, "\circ"] & G \star C \arrow[d, "L^\star_G f"] \\ F \star C' \arrow[r, "R^\star_{C'}\alpha"'] & G \star C' \end{tikzcd} \quad \equiv \quad \begin{tikzcd} FC \arrow[r, "R^\star_C\alpha"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\circ"] & GC \arrow[d, "Gf"] \\ FC' \arrow[r, "R^\star_{C'}\alpha"'] & GC' \end{tikzcd} \tag{6}$$

Thus we can see that *if $\mathcal{E}^{\mathcal{C}}$ is to be a category* whose objects are functors and *if $\star$ is to be a bifunctor*, then *the morphisms* $\mathcal{E}^{\mathcal{C}}$ must exist in correspondence with any subset of the natural transformations in $\mathcal{E}$ which includes the identity natural transformations of every functor in $\mathcal{E}^{\mathcal{C}}$. We are free to pick the maximal such category, and (abusively) suppress the $R^\star$ notation, to claim that the morphisms of $\mathcal{E}^{\mathcal{C}}$ *are* the natural transformations between its functors. That is, rather than writing $R^\star_C\alpha$ we will now write αC; some texts use the notation α_C to reflect the alternate characterization of natural transformations as object-indexed collections of arrows.

2.1 Composition of Nat. Trans

This has always confused me, so here's an excellent opportunity to expando the notation and hopefully make some things clearer. Frustratingly, **F for Functor** uses $\cdot$ for ordinary composition (while standard notation is $\circ$) and $\circ$ for another composition operator on nat. trans. Here we use $\circ$ and $\bigcirc$.

We begin by considering a bifunctor which composes functors, called $- \bigcirc -$. It is an object of the (visually intimidating) category $(\mathcal{E}^{\mathcal{C}})^{\mathcal{E}^{\mathcal{D}} \times \mathcal{D}^{\mathcal{C}}}$. Adopting and extending the paper's naming scheme, let

- $F, H, P \in \mathcal{D}^{\mathcal{C}}$; $\alpha \in \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(F, H)$; $\alpha' \in \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(H, P)$;
- $G, K, Q \in \mathcal{E}^{\mathcal{D}}$; $\beta \in \mathrm{Hom}_{\mathcal{E}^{\mathcal{D}}}(G, K)$; $\beta' \in \mathrm{Hom}_{\mathcal{E}^{\mathcal{D}}}(K, Q)$;
- $C, C' \in \mathcal{C}$; $f \in \mathrm{Hom}_{\mathcal{C}}(C, C')$; $D, D' \in \mathcal{D}$; and $g \in \mathrm{Hom}_{\mathcal{D}}(D, D')$.

The behavior of $\bigcirc$ is given as follows:

- $(G \bigcirc F)C = G \star (FC) = GFC \in \mathcal{E}^{\mathcal{C}}$ (i.e. functor composition)
- $(G \bigcirc \alpha)C = (L_G^{\bigcirc}\alpha)C = G(R_C^{\star}\alpha) = G(\alpha C) \in \mathcal{E}$
- $(\beta \bigcirc F)C = (R_F^{\bigcirc}\beta)C = R^{\star}_{L_F^{\star}C}\beta = \beta(L_F^{\star}C) = \beta(FC) \in \mathcal{E}$

The paper asserts that "the coherence conditions follow from naturality", i.e. that $\forall_C$

$$\begin{tikzcd}
(G \bigcirc F)C \arrow[r, "R_F^{\bigcirc}\beta"] \arrow[d, "L_G^{\bigcirc}\alpha"'] \arrow[dr, phantom, "\circ"] & (K \bigcirc F)C \arrow[d, "L_K^{\bigcirc}\alpha"] \\
(G \bigcirc H)C \arrow[r, "R_H^{\bigcirc}\beta"'] & (K \bigcirc H)C
\end{tikzcd}
\equiv
\begin{tikzcd}
GFC \arrow[r, "\beta(FC)"] \arrow[d, "G(\alpha C)"'] \arrow[dr, phantom, "\circ"] & KFC \arrow[d, "K(\alpha C)"] \\
GHC \arrow[r, "\beta(HC)"'] & KHC
\end{tikzcd}$$

This indeed follows from the naturality of β (not α!). So $\beta \bigcirc \alpha \in \mathrm{Hom}_{\mathcal{E}^{\mathcal{C}}}(G \bigcirc F, K \bigcirc H)$ is well-defined.

If we just write down everything we know (a popular technique for earning sympathy on exams), we first get these two "vertical composition" diagrams (in $\mathcal{D}$ and $\mathcal{E}$, respectively; the use of $\circ$ takes place in $\mathcal{D}^{\mathcal{C}}$ on the left and $\mathcal{E}^{\mathcal{D}}$ on the right):

$$\begin{tikzcd}
FC \arrow[r, "Ff"] \arrow[d, "\alpha C"'] \arrow[dd, bend right=60, "(\alpha' \circ \alpha)C"'] & FC' \arrow[d, "\alpha C'"] \arrow[dd, bend left=60, "(\alpha' \circ \alpha)C'"] \\
HC \arrow[r, "Hf"] \arrow[d, "\alpha' C"'] & HC' \arrow[d, "\alpha' C'"] \\
PC \arrow[r, "Pf"'] & PC'
\end{tikzcd}
\qquad
\begin{tikzcd}
GD \arrow[r, "Gg"] \arrow[d, "\beta D"'] \arrow[dd, bend right=60, "(\beta' \circ \beta)D"'] & GD' \arrow[d, "\beta D'"] \arrow[dd, bend left=60, "(\beta' \circ \beta)D'"] \\
KD \arrow[r, "Kg"] \arrow[d, "\beta' D"'] & KD' \arrow[d, "\beta' D'"] \\
QD \arrow[r, "Qg"'] & QD'
\end{tikzcd}
\tag{7}$$

We also get this "horizontal composition" diagram in $\mathcal{E}$; on the left is the diagram using $\bigcirc$ and on the right is a version with all $\bigcirc$ evaluated.

$$\begin{tikzcd}
(G \bigcirc F)C \arrow[r, "(\beta \bigcirc \alpha)C"] \arrow[d, "(G \bigcirc F)f"'] \arrow[rr, bend left, "((\beta' \circ \beta) \bigcirc (\alpha' \circ \alpha))C"] & (K \bigcirc H)C \arrow[r, "(\beta' \bigcirc \alpha')C"] \arrow[d, "(K \bigcirc H)f"'] & (Q \bigcirc P)C \arrow[d, "(Q \bigcirc P)f"'] \\
(G \bigcirc F)C' \arrow[r, "(\beta \bigcirc \alpha)C'"'] \arrow[rr, bend right, "((\beta' \circ \beta) \bigcirc (\alpha' \circ \alpha))C'"'] & (K \bigcirc H)C' \arrow[r, "(\beta' \bigcirc \alpha')C'"'] & (Q \bigcirc P)C'
\end{tikzcd}
\qquad
\begin{tikzcd}
GFC \arrow[r, "K(\alpha C) \circ \beta(FC)"] \arrow[d, "GFf"'] \arrow[rr, bend left, "Q((\alpha' \circ \alpha)C) \circ (\beta' \circ \beta)(FC)"] & KHC \arrow[r, "Q(\alpha C') \circ \beta(KC)"] \arrow[d, "KHf"'] & QPC \arrow[d, "QPf"'] \\
GFC' \arrow[r, "K(\alpha C') \circ \beta(FC')"'] \arrow[rr, bend right, "Q((\alpha' \circ \alpha)C') \circ (\beta' \circ \beta)(FC')"'] & KHC' \arrow[r, "Q(\alpha' C') \circ \beta'(KC')"'] & QPC'
\end{tikzcd}
\tag{8}$$

The rectangles commute by definition of natural transformations while the upper and lower faces commute by bifunctorality of $\bigcirc$ (namely, that it preserves composition).